Method of Resolving Functions in the Theory of Conflict—Controlled Processes

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Abstract The paper is devoted to investigation of game problems on bringing a trajectory of dynamic system to a cylindrical terminal set. We proceed with representation of a trajectory of dynamic system in the form, in which the block of initial data is separated from the control block. This makes it feasible to consider a wide spectrum of functional-differential systems. The method of resolving functions, based on use of the inverse Minkovski functionals, serves as ideological tool for study. Attention is focused on the case when Pontryagin's condition does not hold. In this case the upper and lower resolving functions of two types are introduced. With their help sufficient conditions of approach a terminal set in a finite time are deduced. Various method schemes are provided and comparison with Pontryagin's first direct method is given. Efficiency of suggested mathematical scheme is illustrated with a model example.

1 Introduction

There are a number of fundamental methods, which reveal structure of game problems and provide mathematical constructions for analysis of conflict situations. First of all these are Pontryagin's methods (first direct method and method of alternated integral) [1], the extreme targeting rule of Krasovskii [2], the method of Isaacs [3], related with the dynamic programming and expressed in the form of functional relationships, the method of T_ε-operators of Pshenichnyi [4], which employs ideas of alternated integral, and is, in essence, is a reflection of dynamic programming ideas in the form of set-valued mappings. The method of resolving functions [5],

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Y. P. Kondratenko et al. (eds.), *Advanced Control Techniques in Complex*

created in recent years, can be also assigned to these methods. Various problems of control in conditions of uncertainty are explored in [6–12].

It should be noted that the extreme targeting rule and method of resolving functions theoretically substantiate, known from practice, the method of pursuit along the Euler line of sight and the rule of pursuit along a ray, in particular, "parallel pursuit" rule, respectively. The game methods have found wide application [13–15]. The method of resolving functions can be applied to study of the impulse processes on the basis of results of the monograph [16] and of functional-differential systems [17].

This paper is dedicated to development of the method of resolving functions. An attractive feature of this method is that it allows efficiently apply modern technique of set-valued mappings and their selections to substantiate the game constructions and to obtain on their basis rich in content results. This method was originated during analysis of the game problems of pursuit-evasion, in particular, as a consequence of solving the problem of escape of single controlled object from a group of pursuers.

In 1969 L. S. Pontryagin formulated and solved in linear case the problem of avoiding meeting one controlled object with another one, starting from arbitrary initial states and on all half-infinite interval of time [1]. Later on, at the conference on game theory (1974) the statement and solution of the problem of avoiding meeting single player with a group of pursuers was independently presented by Gusyatnikov and Chikrii [18]. In so doing, the different methods were used: the manoeuvre of turning movement [19]—by the former and the method of escape along direction by the latter [20].

In the papers [18, 20] the following function was introduced:

$$\omega(n,\nu) = \min_{\|p_i\|=1} \max_{\|v\|=1} \min_{i=1,\ldots,k} |(p_i, v)|,\ p_i, v \in R^n, n, k \geq 2, 0 \leq \omega(n,\nu) \leq 1 \quad (1)$$

It was destined to play a key role in creating the method of resolving functions. Function $\omega(n,\nu)$ determines an advantage of the evader in control resources upon each of the pursuers, which is suffice to rectilinearly escape, starting from arbitrary initial states.

Consider the game of group pursuit with exact capture as a goal:

$$\dot{x}_i = u_i,\ \|u_i\| \leq 1,\ i = 1, \ldots, k,\ x_i(0) = x_i^0,\ x_i \in R^n,$$
$$\dot{y} = v,\ \|v\| \leq 1,\ y(0) = y^0,\ y \in R^n, n \geq 2. \quad (2)$$

Removing external minimum and modulo and setting $p_i = \frac{x_i^0 - y_0}{\|x_i^0 - y_0\|}$ in (1) we come to the assertion: $y_0 \in \mathrm{int}co\{x_i^0\}$ if and only if $\max_{\|v\|=1} \min_{i=1,\ldots,k} (p_i, v) > 0$, whence follows the main result: for feasibility of capture in the group pursuit problem (2) it is necessary and sufficient that

$$y_0 \in \mathrm{int}co\{x_i^0\}$$

This fact was noticed by B. N. Pshenichnyi when seeing the function (1) [21]. In the proof of this result the pursuers apply the rule of parallel pursuit and their controls appear as counter-controls [2] which depend on the initial players' states and certain scalar function, the latter being the greatest positive root of special quadratic equation. This function is a prototype of the resolving function.

The full description of the method of resolving functions is contained in the monograph [5] and it's modification for more general dynamics—in the paper [22]. The problems of group pursuit are also considered in the books [23, 24].

This paper is devoted to the problem of pursuit with single pursuer and a single evader in the case when Pontryagin's condition (the condition for advantage of the pursuer over the evader) does not hold. The upper and the lower resolving functions of two types are introduced that makes it possible to essentially broaden the class of problems solvable on the basis of the above mentioned ideology.

2 Problem Statement and Players Goals

Let us consider in finite-dimensional Euclidian space R^n, $n \geq 2$, conflict-controlled process whose evolution is given by the equation

$$z(t) = g(t) + \int_0^t \Omega(t,\tau)\phi(u(\tau), v(\tau))d\tau, t \geq 0 \quad (3)$$

where $z(t) \in R^n$, the function of initial data $g(t)$, $g : R_+ \to R^n$, $R_+ = \{t : t \geq 0\}$, is the Lebesgue measurable and bounded for $t > 0$, the matrix function $\Omega(t,\tau)$, $t \geq \tau \geq 0$, is bounded almost everywhere, measurable in t and summable in τ for every $t \in R_+^{\prime}$. Control unit includes function $\phi(u, v)$, $\varphi : U \times V \to R^n$, which is assumed to be jointly continuous in its variables on direct product of nonempty compacts U and V, $U \in K(R^m)$, $V \in K(R^l)$, n, m, l are natural integers.

Admissible controls of the players, $u(\tau)$, $u : R_+ \to U$, $v(\tau)$, $v : R_+ \to V$, are measurable functions of time.

Let the terminal set M^*, having the cylindrical form be given

$$M^* = M_0 + M \quad (4)$$

where M_0 is linear subspace from R^n, a $M \in K(L)$, $L = M_0^{\perp}$ is orthogonal complement to M_0 in R^n.

Goals of the players are opposite, moreover, the performance criterion is the time. The first one (u) tries in the shortest time to bring a trajectory (3) to the terminal set (4), and the second one (v) tries to delay maximally the instant of trajectory hitting the set M^* or avoid the hitting. General representation of solution of dynamical system in the form (3) makes it possible to consider in the frames of unified scheme a wide range of quasilinear functional-differential systems, functioning under the condition of conflict, in particular, systems of ordinary differential, integral-differential and differential-difference equations as well as systems of equations with classical fractional derivatives of Rienmann–Liouville, regularized fractional derivatives of Dzhirbashian–Nersesian–Caputo, sequential fractional derivatives of Miller-Ross, the generalized Hilfer derivatives [25], impulse systems [26]. Similar representation in discrete case makes it feasible to study multistep processes, in particular, the discrete systems of fractional order by Grunwald–Letnikov [27] and the hybrid systems.

Specific forms of the function $g(t)$ and the matrix function $\Omega(t, \tau)$ define the type of conflict-controlled process.

Let us take the side of the first player. If the game (3)–(4) is evolving on the interval $[0, T]$, then we select control at time instant t in the form of measurable function

$$u(t) = u(g(T), v_t(\cdot)), u(t) \in U, t \in [0, T] \tag{5}$$

where $v_t(\cdot) = \{v(s) : s \in [0, t]\}$ is a prehistory of admissible control of the second player until the time instant t or, in the form of counter-control,

$$u(t) = u(g(T), v(t)), u(t) \in U, t \in [0, T]. \tag{6}$$

In the case (5) we speak about quasistrategy of the first player, in the case (6)—about Hajek stroboscopic strategy [11], which prescribes counter-control by Krasovskii [2].

Objective of the paper is to establish sufficient conditions for termination of approach game (3)–(4) in certain guaranteed time, using the second player control prehistory, as well as in the class of stroboscopic strategies by comparing guaranteed time of game termination with guaranteed time of the Pontryagin's first direct method [1, 12].

3 Pontryagin's First Direct Method

Let us denote by π the operator of orthogonal projecting from R^n into L. We set

$$\varphi(U, v) = \{\varphi(u, v) : u \in U\}$$

and introduce set-valued mapping

$$W(t, \tau, v) = \pi\Omega(t, \tau)\varphi(U, v)$$

on the set $\Delta \times V$, where

$$\Delta = \{(t, \tau) : 0 \le \tau \le t < +\infty\}$$

is a plane cone.

Condition 1 Mapping $W(t, \tau, v)$ is closed-valued on the direct product of cone Δ and compact V.

Let us consider

$$W(t, \tau) = \bigcap_{v \in V} W(t, \tau, v), 0 \le \tau \le t < +\infty,$$

and denote [28]

$$\operatorname{dom} W = \{(t, \tau) \in \Delta : W(t, \tau) \neq \emptyset\}.$$

Pontryagin's condition. $\operatorname{dom} W = \Delta$.

If Condition 1 and Pontryagin's condition hold, then mapping $W(t, \tau)$ is closed-valued and measurable in τ, $\tau \in [0, t]$. Therefore [29], there exists at least one measurable in τ selection, namely, Pontryagin's selection.

Let us consider Pontryagin's function, under the above mentioned conditions,

$$p(g(\cdot)) = \inf\{t \ge 0 : t \in P(g(\cdot))\},$$

where

$$P(g(\cdot)) = \{t \ge 0 : \pi g(t) \in M - \int_0^t W(t, \tau) d\tau\}. \tag{7}$$

Here by integral of set-valued mapping is meant Aumann's integral [30]. If in (7) the inclusion in braces does not hold for all t, $t \ge 0$, then we set

$$P(g(\cdot)) = \emptyset, \text{ and } p(g(\cdot)) = +\infty.$$

Theorem 1 *Let for conflict-controlled process* (3)–(4) Condition 1 and *Pontryagin's condition hold, and* $P \in P(g(\cdot)) \neq \emptyset$.

Then a trajectory of the process (3) *can be brought into the terminal set* (4) *at instant* P *by means of the Hajek stroboscopic strategy, which prescribes the corresponding Krasovskii' counter-control.*

Proof From the assumptions of Theorem 1 and the inclusion in the relation (7) there follows that

$$\pi g(P) \in M - \int_0^P W(P, \tau) d\tau.$$

This means that there exist a point m, $m \in M$, and, by definition of Aumann's integral, Pontryagin's measurable selection $\gamma(P,\tau)$, $\gamma(P,\tau) \in W(P,\tau)$, $\tau \in [0,P]$, such that

$$\pi g(P) = m - \int_0^P \gamma(P,\tau)d\tau. \tag{8}$$

Let us consider the set-valued mapping

$$U_0(\tau, v) = \{u \in U : \pi\Omega(P,\tau)\phi(u,v) - \gamma(P,\tau) = 0\},\ \tau \in [0,P],\ v \in V$$

The mapping $U_0(\tau, v)$ is closed-valued and $L \times B$-measurable [22, 31]. Therefore, by the theorem on measurable choice [29], there exists at least one $L \times B$-measurable selection $u_0(\tau, v)$, $u_0(\tau, v) \in U_0(\tau, v)$, which appears as superpositionally measurable function [22]. Let us select control of the first player in the form of measurable function $u(\tau) = u_0(\tau, v(\tau))$, $\tau \in [0,P]$, where $v(\tau)$ is an admissible control of the second player.

From the representation (3), in view of the relation (8) and equality in the expression for $U_0(\tau, v)$, it immediately follows it that $\pi g(P) = m \in M$.

Remark The above provided scheme is usually called the first Pontryagin direct method and the time $p(g(\cdot))$ is the minimal guaranteed time of this scheme. The suggested proof differs from the original one [1] and is a generalization of the result of P. B. Gusyatnikov and M. S. Nikolskij, dealing with the mentioned method for linear differential games and using Filippov-Castaing theorem on measurable choice [32].

In next section we do not assume that Pontryagin condition holds and focus our investigations on development of the ideology of resolving functions [5, 22] in this case.

4 Schemes of the Method of Resolving Functions

Let $\gamma(t,\tau)$, $\gamma : \Delta \to L$, be some function, which is almost everywhere bounded, measurable in t and summable in τ, $\tau \in [0,t]$ for each $t \in R_+$. In the sequel it will be called the shift function.

Let us denote

$$\xi(t) = \xi(t, g(t), \gamma(t,\cdot)) = \pi g(t) + \int_0^t \gamma(t,\tau)d\tau$$

and consider the set-valued mapping

$$\mathrm{A}(t,\tau,v) = \{\alpha \geq 0 : [\pi\Omega(t,\tau)\phi(u,v) - \gamma(t,\tau)] \cap \alpha[M - \xi(t)] \neq \emptyset\}, \tag{9}$$

$$\mathrm{A} : \Delta \times V \to 2^{R_+}.$$

Condition 2 Set-valued mapping $\mathrm{A}(t,\tau,v)$ has nonempty images on the set $\Delta \times V$.

Under this condition, we consider scalar upper and lower resolving functions of the first type

$$\alpha^*(t,\tau,v) = sup\{\alpha : \alpha \in \mathrm{A}(t,\tau,v)\},$$
$$\alpha_*(t,\tau,v) = inf\{\alpha : \alpha \in \mathrm{A}(t,\tau,v)\}.$$

If, in addition, Pontryagin's condition holds, then, to emphasize the role of the Minkowski functional [28, 33] and its inverse in the method scheme, we represent the upper resolving function $\alpha^*(t,\tau,v)$ in another way. To this end, we introduce the inverse Minkowski functional [5] for closed set X, $X \subset R^n$, $0 \in X$:

$$\alpha_X(p) = sup\{\alpha \geq 0 : \alpha p \in X\},\ p \in R^n.$$

Then [22] $\alpha^*(t,\tau,v) = \sup_{m \in M} \alpha_{W(t,\tau,v)-\gamma(t,\tau)}(m - \xi(t))$.

Since image of the mapping $\mathrm{A}(t,\tau,v)$ present itself numerical sets on the semi-axis R_+, then the upper resolving function is the support function of this mapping in the direction +1. Taking into account properties of the conflict-controlled process (3), (4), Conditions 1 and 2, and the theorem on characterization and inverse image [29], we can show [22, 31] that closed-valued mapping $\mathrm{A}(t,\tau,v)$ is jointly $L \times B$-measurable in (τ, v), $\tau \in [0,t]$, $v \in V$, and the upper and the lower resolving functions are jointly $L \times B$-measurable in (τ, v) by virtue of the theorem on support function [29].

Let $V(\cdot)$ be a set of measurable functions $v(\tau)$, $\tau \in [0,+\infty)$, taking their values in V. Since at fixed t the function $\alpha^*(t,\tau,v)$ is jointly $L \times B$-measurable in (τ, v), then it is superpositionally measurable [22], i.e. $\alpha^*(t,\tau,v(\tau))$ is measurable in τ, $\tau \in [0,t]$, for each measurable function $v(\cdot) \in V(\cdot)$. The lower resolving function possesses the same property.

The upper resolving function $\alpha^*(t,\tau,v)$, in essence, reflects maximal gain of the first player in the game at the time instant τ on interval $[0,t]$ under instantaneous counter-action v. We associate with this function the set

$$T(g(\cdot),\gamma(\cdot,\cdot)) = \{t > 0 : \inf_{v(\cdot) \in V(\cdot)} \int_0^t \alpha^*(t,\tau,v(\tau))d\tau \geq 1\} \tag{10}$$

and its least element

$$t(g(\cdot),\gamma(\cdot,\cdot)) = \inf\{t : t \in T(g(\cdot),\gamma(\cdot,\cdot))\}.$$

If for some t, $t > 0$, $\alpha^*(t, \tau, v) = +\infty$, $\tau \in [0, t]$, $v \in V$, then in this case it is natural to set the value of integral in the relation (10) to be equal to $+\infty$. Then the corresponding inequality is readily satisfied and $t \in T(g(\cdot), \gamma(\cdot, \cdot))$. In case the inequality in (10) does not hold for all $t > 0$, we set $T(g(\cdot), \gamma(\cdot, \cdot)) = \emptyset$ and correspondingly $t(g(\cdot), \gamma(\cdot, \cdot)) = +\infty$.

Let us denote

$$\mathrm{A}(t, \tau) = \bigcap_{v \in V} \mathrm{A}(t, \tau, v), (t, \tau) \in \Delta.$$

Condition 3 The set-valued mapping $\mathrm{A}(t, \tau)$ has nonempty image on the cone Δ.

Under this condition we introduce upper and lower resolving functions of the second type

$$\alpha^*(t, \tau) = \sup\{\alpha : \alpha \in \mathrm{A}(t, \tau)\},$$
$$\alpha_*(t, \tau) = \inf\{\alpha : \alpha \in \mathrm{A}(t, \tau)\}.$$

Lemma Let conditions 1 and 3 hold for the conflict-controlled process (3)–(4), the mapping $\mathrm{A}(t, \tau, v)$ be compact-valued, and the upper resolving function $\alpha^*(t, \tau, v)$ be bounded on the set $\Delta \times V$. Then the following inequality is true:

$$\inf_{v \in V} \alpha^*(t, \tau, v) \geq \alpha^*(t, \tau), (t, \tau) \in \Delta. \tag{11}$$

If, in addition, the mapping $\mathrm{A}(t, \tau, v)$, $(t, \tau) \in \Delta$, $v \in V$, is convex-valued, then (11) turns into equality.

Proof By construction, the studied functions have the forms

$$\inf_{v \in V} \alpha^*(t, \tau, v) = \inf_{v \in V} \sup\{\alpha : \alpha \in \mathrm{A}(t, \tau, v)\},$$

$$\alpha^*(t, \tau) = \sup_{v \in V}\{\alpha : \alpha \in \cap \mathrm{A}(t, \tau, v)\}, 0 \leq \tau \leq t < +\infty.$$

Let us denote $\alpha^* = \alpha^*(t, \tau)$. Since the mapping $\mathrm{A}(t, \tau, v)$ is compact-valued, then $\mathrm{A}(t, \tau)$ is compact-valued in α^*, $\alpha^* \in \mathrm{A}(t, \tau, v)$, for each $v \in V$. This implies

$$\alpha^* \leq \sup\{\alpha : \alpha \in \mathrm{A}(t, \tau, v)\} \quad \forall v \in V,$$

therefore

$$\alpha^* \leq \inf_{v \in V} \sup\{\alpha : \alpha \in \mathrm{A}(t, \tau, v)\} = \inf_{v \in V} \alpha^*(t, \tau, v).$$

From the assumption that mapping $\mathrm{A}(t, \tau, v)$ is convex-valued and compact-valued there follows that on the set $\Delta \times V$

$$\mathrm{A}(t, \tau, v) = [\alpha_*(t, \tau, v), \alpha^*(t, \tau, v)].$$

Then the nonemptiness of the images of mapping $\mathrm{A}(t, \tau)$, $(t, \tau) \in \Delta$, means that $\mathrm{A}(t, \tau) = [\alpha_*(t, \tau), \alpha^*(t, \tau)]$, moreover

$$\alpha_*(t, \tau) = \sup_{v \in V} \alpha_*(t, \tau, v) \leq \inf_{v \in V} \alpha^*(t, \tau, v) = \alpha^*(t, \tau), (t, \tau) \in \Delta.$$

Thereby equality in the relation (11) is proved.

Let us introduce into consideration the numerical functions

$$\alpha^*(t) = \int_0^t \alpha^*(t, \tau) d\tau, \alpha_*(t) = \int_0^t \alpha_*(t, \tau) d\tau,$$

$$\delta(g(\cdot), \gamma(\cdot, \cdot)) = \inf\{t : t \in \Theta(g(\cdot), \gamma(\cdot, \cdot))\},$$

where

$$\Theta(g(\cdot), \gamma(\cdot, \cdot)) = \{t > 0 : \alpha^*(t) \geq 1\}.$$

Theorem 2 *Let for the conflict-controlled process* (3)–(4) *Conditions* 1, 3 *hold, in addition* $M = \mathrm{co}M$, *moreover, for the given function* $g(\cdot)$ *and the shift* $\gamma(\cdot, \cdot)$

$$T \in T(g(\cdot), \gamma(\cdot, \cdot)) \neq \emptyset.$$

Then, if $\alpha_*(T) < 1$ *then a trajectory of the process* (3) *can be brought to the terminal set* (4) *at the time instant* T, *by means of control of the form* (5), *if, otherwise,* $\alpha^*(T) \geq 1$, *then by means of counter-control, for arbitrary admissible counteractions of the second player.*

Proof Let $v(\tau)$, $v : [0, T] \to V$ be a measurable function. Let us assume that $\alpha^*(T, \tau, v) \neq +\infty$, $\tau \in [0, T]$, $v \in V$.

We consider the testing function

$$h(t) = 1 - \int_0^t \alpha^*(T, \tau, v(\tau)) d\tau - \int_t^T \alpha_*(T, \tau) d\tau, \tau \in [0, T].$$

As it was mentioned above, the function $\alpha^*(t, \tau, v)$ is $L \times B$-measurable in (τ, v), $\tau \in [0, T]$, $v \in V$, therefore, it is superpositionally measurable and, consequently, $\alpha^*(T, \tau, v(\tau))$ is a function, measurable in τ, $\tau \in [0, T]$. The function $\alpha_*(T, \tau)$ is also measurable in τ. Hence, $h(t)$ is absolutely continuous on the interval $[0, T]$. Since

$$h(0) = 1 - \int_0^T \alpha_*(T,\tau)d\tau = 1 - \alpha_*(T) > 0,$$

and, by definition of moment of T,

$$h(T) = 1 - \int_0^T \alpha^*(T,\tau,v(\tau))d\tau \le 0,$$

then, by the well-known theorem of mathematical analysis, there exists a time instant t_*, $t_* = t(v(\cdot))$, $t_* \in [0,T]$, such that $h(t_*) = 0$. Note that the instant of switching t_* depends on control prehistory of the second player $v_{t_*}(\cdot)$.

We shall call the time intervals $[0,t_*)$ and $[t_*,T]$ "active" and "passive", respectively. Let us describe how the first player chooses its control on each of them. To this end, let us consider the compact-valued mappings

$$\begin{aligned} U_1(\tau,v) &= \{u \in U : \pi\Omega(T,\tau)\phi(u,v) - \gamma(T,\tau) \in \\ &\in \alpha^*(T,\tau,v)[M - \xi(T)]\}, \quad \tau \in [0,t_*) \\ U_2(\tau,v) &= \{u \in U : \pi\Omega(T,\tau)\phi(u,v) - \gamma(T,\tau) \in \\ &\in \alpha_*(T,\tau)[M - \xi(T)]\}, \quad \tau \in [t_*,T] \end{aligned} \tag{12}$$

From construction of the mappings $\mathsf{A}(T,\tau,v)$ and $\mathsf{A}(T,\tau)$ there follows that $U_i(\tau,v)$, $i = 1,2$, have nonempty images.

By virtue of the theorem on inverse image [29] the set-valued mappings $U_1(\tau,v)$, $U_2(\tau,v)$ are $L \times B$-measurable [30], therefore, by the theorem on measurable choice [29], in both there exist at least one $L \times B$- measurable selection $u_1(\tau,v)$, $u_2(\tau,v)$, which are superpositionally measurable functions.

Let us denote $u_1(\tau) = u_1(\tau,v(\tau))$, $u_2(\tau) = u_2(\tau,v(\tau))$.

We set control of the first player on the "active" interval to be equal to $u_1(\tau)$, and on the "passive" one-to $u_2(\tau)$. Thus, despite the fact that the first player on each of the interval does not use directly prehistory of control of the second player but only its instantaneous control, in order to determine the switching instant t_* the prehistory is required .

Under chosen controls, we have from representation (3)

$$\pi z(T) = \pi g(T) + \int_0^T \pi\Omega(T,\tau)\phi(u(\tau),v(\tau))d\tau$$

$$= \pi g(T) + \int_0^{t_*} \pi\Omega(T,\tau)\phi(u_1(\tau),v(\tau))d\tau$$

$$+ \int_{t_*}^T \pi\Omega(T,\tau)\phi(u_2(\tau),v(\tau))d\tau \pm \int_0^T \gamma(T,\tau)d\tau$$

Using the relations (12), from above formula we deduce

$$\pi z(T) \in \pi g(T) + \int_0^{t_*} \alpha^*(T,\tau,v(\tau))[M - \xi(T)]d\tau$$

$$+ \int_{t_*}^T \alpha_*(T,\tau)[M - \xi(T)]d\tau + \int_0^T \gamma(T,\tau)d\tau$$

$$= \xi(T)(1 - \int_0^{t_*} \alpha^*(T,\tau,v(\tau))d\tau - \int_{t_*}^T \alpha_*(T,\tau)d\tau)$$

$$+ \int_0^{t_*} \alpha^*(T,\tau,v(\tau))Md\tau + \int_{t_*}^T \alpha_*(T,\tau)Md\tau$$

$$= [\int_0^{t_*} \alpha^*(T,\tau,v(\tau))d\tau + \int_{t_*}^T \alpha_*(T,\tau)d\tau]M = M.$$

Here the equality $h(t_*) = 0$ is twice taken into account, and transition of set-valued mappings with set M can be substituted by usage of the support functions technique [28, 34].

As follows from the expression (10), the case $\alpha^*(T,\tau,v) = +\infty$ for some $\tau \in [0,T]$, $v \in V$, is possible only under the conditions

$$0 \in M - \xi(T), 0 \in \Omega(T,\tau)\phi(U,v) - \gamma(T,\tau).$$

It is evident that in this case

$$\mathsf{A}(T,\tau,v) = [0,+\infty), \tau \in [0,T], v \in V,$$

and $\alpha_*(T,\tau) = 0$, $\tau \in [0,T]$.

This makes it possible to select as resolving functions at the points $\tau \in [0,T]$, where $\alpha^*(T,\tau,v(\tau)) = +\infty$, an arbitrary finite, superpositionally measurable function, which takes its values on the semi-infinite interval $[0,+\infty)$ with the only condition that final resolving function on the interval $[0,T]$ provides the relation $h(t_*) = 0$ for certain switching instant t_*, $t_* \in [0,T]$. Thereby construction of control on "active" and "passive" intervals is reduced to the previous case.

The case when $\alpha^*(T, \tau, v) = +\infty$ for all $\tau \in [0, T]$, $v \in V$, corresponds to the first Pontryagin method [1, 12]. Indeed, inclusion

$$0 \in \Omega(T, \tau)\phi(U, v) - \gamma(T, \tau) \quad \forall \tau \in [0, T], \quad v \in V,$$

provides non-emptiness of the set of images of Pontryagin's mapping $W(T, \tau)$ on the game interval $[0, T]$, and the shift function $\gamma(T, \tau)$ appears as measurable selection of the mapping $W(T, \tau)$, i.e., the Pontryagin selection.

The inclusion $0 \in M - \xi(T)$ implies the relation

$$\pi g(T) \in M - \int_0^T W(T, \tau)d\tau,$$

from which, by virtue of Theorem 1, there follows that the game (3)–(4) can be terminated in the class of stroboscopic strategies.

Let us consider separately the case $\alpha^*(T) \geq 1$, $\alpha_*(T) < 1$. We introduce the testing function

$$h_1(t) = 1 - \int_0^t \alpha^*(T, \tau)d\tau - \int_t^T \alpha_*(T, \tau)d\tau.$$

It is natural to examine only the case $\alpha^*(T, \tau) \neq +\infty$, $\tau \in [0, T]$. Then

$$h_1(0) = 1 - \alpha_*(T) > 0, h_1(T) = 1 - \alpha^*(T) \leq 0,$$

and, by virtue of continuity of the function $h_1(t)$, there exists a moment $t_*^1, t_*^1 \in [0, T]$, such that $h_1(t_*^1) = 0$. It should be noted that the time instant t_*^1 does not depend on $v(\cdot)$ in this case.

We consider set-valued mappings (12) on "active" and "passive" intervals $[0, t_*^1)$ and $[t_*^1, T]$, with the function $\alpha^*(T, \tau)$ replaced for $\alpha^*(T, \tau, v)$ in the expression for $U_1(\tau, v)$. Using the property for $L \times B$-measurability of compact-valued mappings $U_1(\tau, v)$, $U_2(\tau, v)$, we select in them $L \times B$-measurable selections, which determine admissible controls on both intervals. Final considerations are similar to the conclusions in previous case.

Let us introduce into consideration the functions

$$\alpha(t) = \int_0^t \inf_{v \in V} \alpha^*(t, \tau, v)d\tau,$$

$$\alpha(t, \tau) = \frac{1}{\alpha(t)} \inf_{v \in V} \alpha^*(t, \tau, v), 0 \leq \tau \leq t < +\infty.$$

Condition 4 For the chosen shift function $\gamma(t, \tau)$, $(t, \tau) \in \Delta$, the function $\inf_{v \in V} \alpha^*(t, \tau, v)$ is measurable in τ, $\tau \in [0, t]$, $t > 0$, and

$$\inf_{v(\cdot) \in V(\cdot)} \int_0^t \alpha^*(t, \tau, v)d\tau = \int_0^t \inf_{v \in V} \alpha^*(t, \tau, v)d\tau, t > 0.$$

Assumptions on the function $\alpha^*(t, \tau, v)$, ensuring fulfillment of the above equality are analyzed, for example, in [31].

Theorem 3 *Let for the game problem (3)–(4) with some shift function $\gamma(t, \tau)$, $(t, \tau) \in \Delta$, $T \in T(g(\cdot), \gamma(\cdot, \cdot)) \neq \emptyset$, Conditions 1, 3, 4 hold and the mapping $A(t, \tau, v)$ be convex-valued on the set $\Delta \times V$, $M = \mathrm{co}M$. Let us suppose that the following inequality is true:*

$$\alpha(T, \tau) \geq \sup_{v \in V} \alpha_*(T, \tau, v), \tau \in [0, T]. \tag{13}$$

Then a trajectory of the process (3) can be brought to the set (4) at the time-instant T with help of certain counter-control.

Proof It suffices to consider the case $\alpha^*(T, \tau, v) < +\infty$, $\tau \in [0, T]$, $v \in V$. Since, by virtue of the inequality in (4.2), $\alpha(T) \geq 1$, then

$$\alpha(T, \tau) = \frac{1}{\alpha(T)} \inf_{v \in V} \alpha^*(T, \tau, v) \leq \inf_{v \in V} \alpha^*(T, \tau, v), \tau \in [0, T],$$

and, thereby,

$$\alpha(T, \tau) \leq \alpha^*(T, \tau, v) \quad \forall v \in V, \tau \in [0, T].$$

Taking into account the inequality (13), one can draw conclusion that $\alpha(T, \tau) \in A(T, \tau, v)$ for $v \in V$, $\tau \in [0, T]$, and, consequently, $\alpha(T, \tau) \in \mathsf{A}(T, \tau)$, $\tau \in [0, T]$.

Let us consider the set-valued mapping

$$\begin{aligned} U(\tau, v) = \{u \in U : \pi\Omega(T, \tau)\phi(u, v) - \gamma(T, \tau) \in \\ \in \alpha(T, \tau) \in \alpha(T, \tau)[M - \xi(T)]\}, \tau \in [0, T], v \in V \end{aligned} \tag{14}$$

Analogously to the previous case, the mapping $U(t, \tau)$ is compact-valued and $L \times B$-measurable, therefore, by the theorem on measurable choice [29], there exists a measurable selection $u(t, \tau)$, $u(t, \tau) \in U(t, \tau)$, which appears as superpositionally measurable function. Then, if $v(\cdot)$, $v(\cdot) \in V(\cdot)$, is an arbitrary admissible control of the second player, then we set control of the first player to be equal to

$$u(\tau) = u(\tau, v(\tau)), \tau \in [0, T].$$

From representation (3), with account of the inclusion in (14), we obtain

$$\pi z(T) \in \xi(T)[1 - \int_0^T \alpha(T,\tau)d\tau] + \int_0^T \alpha(T,\tau)Md\tau.$$

Since M is a convex compact, and $\alpha(t,\tau)$, $\tau \in [0,T]$, is a nonnegative function, moreover, $\int_0^T \alpha(T,\tau)d\tau = 1$, then $\int_0^T \alpha(T,\tau)Md\tau = M$ and, therefore, $\pi z(T) \in M$.

5 Scheme with Fixed Points of the Terminal Set Solid Part

One of the assumptions of the above mentioned theorems, concerning the method of resolving functions, is convexity of the set M (solid part of the terminal set). Now we provide one scheme of the method of resolving functions without this assumption, in which the targeting point of the set remains the same as time goes on.

For simplicity sake, we assume that Pontryagin's condition holds. It immediately follows that the lower resolving function equals zero and the upper one coincides with the ordinary resolving function [22]. As previously, in this case $\gamma(t,\tau)$ is a selection of the set-valued mapping $W(t,\tau)$. We fix some point m, $m \in M$, and set $\eta(t,m) = \xi(t) - m$, $t \geq 0$. Let us introduce a set-valued mapping

$$\mathrm{A}(t,\tau,v,m) = \{\alpha \geq 0 : -\alpha\eta(t,m) \in \pi\Omega(t,\tau)\varphi(U,v) - \gamma(t,\tau)\}$$

and its support function in the direction +1

$$\alpha(t,\tau,v,m) = \sup\{\alpha : \alpha \in \mathrm{A}(t,\tau,v,m)\},\ t \geq \tau \geq 0,\ v \in V,\ m \in M.$$

Note that if $\eta(t,m) = 0$ then $\mathrm{A}(t,\tau,v,m) = [0,+\infty)$ for $\tau \in [0,t]$, $v \in V$, and, consequently, $\alpha(t,\tau,v,m) = +\infty$.

Let us consider the set

$$T(g(\cdot),m,\gamma(\cdot,\cdot)) = \{t > 0 : \inf_{v(\cdot)} \int_0^t \alpha(t,\tau,v(\tau),m)d\tau \geq 1\}$$

and its least element

$$t(g(\cdot),m,\gamma(\cdot,\cdot)) = \inf\{t > 0 : t \in T(g(\cdot),m,\gamma(\cdot,\cdot))\}.$$

under assumption that this element exists. If the function $t(g(\cdot),m,\gamma(\cdot,\cdot))$ is lower semi-continuous in m, $m \in M$, then it generates the marginal set

$$M(g(\cdot),\gamma(\cdot,\cdot)) = \{m \in M : t(g(\cdot),m,\gamma(\cdot,\cdot)) = \bar{t}(g(\cdot),\gamma(\cdot,\cdot))\},$$

where

$$\bar{t}(g(\cdot),\gamma(\cdot,\cdot)) = \min_{m \in M} t(g(\cdot),m,\gamma(\cdot,\cdot)).$$

For the sake of comparison, note that [22]

$$t(g(\cdot),\gamma(\cdot,\cdot)) = \inf\{t > 0 : \inf_{v(\cdot)} \int_0^t \sup_{m \in M} \alpha(t,\tau,v(\tau),m)d\tau \geq 1\},$$

$$\bar{t}(g(\cdot),m,\gamma(\cdot,\cdot)) = \inf\{t > 0 : \sup_{m \in M} \inf_{v(\cdot)} \int_0^t \alpha(t,\tau,v(\tau),m)d\tau \geq 1\},$$

where $\sup_{m \in M} \alpha(t,\tau,v,m) = \alpha(t,\tau,v)$.

Theorem 4 *Let for the conflict-controlled process* (3)–(4) *Pontryagin's condition hold, and for given function $g(t)$, some selection $\gamma(t,\tau)$, $\gamma(t,\tau) \in W(t,\tau)$ and point m, $m \in M$,*

$$T_m \in T(g(\cdot),m,\gamma(\cdot,\cdot)) \neq \emptyset.$$

Then the projection of a trajectory (3) *on the subspace L can be brought into the point m at the moment T_m with the help of certain control, prescribed by the quasi-strategy. If, in addition, $m \in M(g(\cdot),\gamma(\cdot,\cdot))$, then it will occur at the moment $\bar{t}(g(\cdot),\gamma(\cdot,\cdot))$.*

Without going into details of the proof we only describe the procedure of control construction. It should be mentioned that the line of reasoning is analogous, for example, to that of the proof of Theorem 2.

If $\eta(T_m,m) \neq 0$, then we divide the interval $[0,T_m]$ into the active and passive parts $[0,t_m)$ and $[t_m,T_m]$, where the moment of switching t_m is a root of the testing function

$$h_m(t) = 1 - \int_0^t \alpha(T_m,\tau,v(\tau),m)d\tau.$$

Then control of the first player on the active part $[0,t_m)$ is chosen in the form of superpositionally measurable selection of the set-valued mapping

$$\mathrm{U}(\tau,v,m) = \{u \in U : \pi\Omega(T_m,\tau)\varphi(u,v) - \gamma(T_m,\tau) = -\alpha(T_m,\tau,v,m)\eta(T_m,m)\},$$

and on the passive part $[t_m, T_m]$—analogously, but with $\alpha(T_m, \tau, v, m) = 0$.

If, otherwise, $\eta(T_m, m) = 0$, then on all interval $[0, T_m]$ control of the first player is chosen in the same way as on the passive part.

6 Connection of the First Direct Method with the Method of Resolving Functions

Let us deduce relations between guaranteed times of the above mentioned methods.

Proposition 1 *Let the game problem (3)–(4) be given. Then for fulfillment of Pontryagin's condition: $W(t,\tau) \neq \emptyset, (t,\tau) \in \Delta$, it is necessary and sufficient that there exists a shift function $\gamma(t,\tau)$ such that*

$$0 \in \mathrm{A}(T, \tau, v) \quad \forall (t,\tau) \in \Delta, v \in V.$$

Proof Let the condition $W(t,\tau) \neq \emptyset$, $(t,\tau) \in \Delta$, hold. Then, by virtue of closed-valuedness and measurability of the mapping $W(t,\tau)$, from the theorem on measurable choice [29] there follows that there exists a measurable selection $\gamma(t,\tau)$, $\gamma(t,\tau) \in W(t,\tau)$. This implies that

$$0 \in W(t,\tau) - \gamma(t,\tau) \quad \forall (t,\tau) \in \Delta$$

or

$$0 \in W(T, \tau, v) - \gamma(t,\tau) \quad \forall (t,\tau) \in \Delta, v \in V.$$

Thereby, zero value of α in expression (9) provides non-emptiness of the sets intersection and, therefore,

$$0 \in \mathrm{A}(T, \tau, v) \quad \forall (t,\tau) \in \Delta, v \in V. \tag{15}$$

Consideration in inverse order results in the required conclusion.

Thus, within the framework of Proposition 1 the shift function $\gamma(t,\tau)$ appears as the Pontryagin selection. We denote the set of such selections by Γ. In addition, we have that

$$0 \in \mathrm{A}(t,\tau) \quad \forall\, (t,\tau) \in \Delta$$

and the corresponding lower resolving functions are

$$\alpha_*(t,\tau,v) = \alpha_*(t,\tau) = 0 \quad \forall (t,\tau) \in \Delta, v \in V. \tag{16}$$

Proposition 2 *Let for some $t > 0$ $W(t,\tau) \neq \emptyset$, $\tau \in [0,t]$. In this case the inclusion*

$$\pi g(t) \in M - \int_0^t W(t,\tau)d\tau \tag{17}$$

holds if and only if there exists a measurable in τ Pontryagin's selection, such that

$$\xi(t, g(t), \gamma(t,\cdot)) \in M.$$

Proof Let the inclusion (17) hold. Then, by definition of the Auman integral there exists a Pontryagin selection, such that

$$\pi g(t) + \int_0^t \gamma(t,\tau)d\tau = \xi(t) \in M. \tag{18}$$

Conversely, if for some Pontryagin's selection the relation (18) is true, then by transferring the integral of selection into the right-hand side of (18) we, all the more, get the inclusion (17).

Thus, if for some time instant t and the chosen Pontryagin's selection the inclusion (18) holds, then

$$\mathrm{A}(t,\tau,v) = [0,+\infty) \quad \forall (t,\tau) \in \Delta, v \in V.$$

Thereby

$$\mathrm{A}(t,\tau) = [0,+\infty), (t,\tau) \in \Delta.$$

Hence, in this case the corresponding upper resolving functions of both types coincide:

$$\alpha^*(t,\tau,v) = \alpha^*(t,\tau) = +\infty, (t,\tau) \in \Delta, v \in V.$$

and the corresponding lower resolving functions equal zero.

Before comparing guaranteed times of the considered methods let us deduce so-called functional form [22] of the first direct method, expressed through special resolving functions.

Let for conflict-controlled process (3), (4) Pontryagin's condition hold. We analyze the set-valued mapping

$$\mathrm{B}(t,\tau) = \{\beta \geq 0 : [W(t,\tau) - \gamma(t,\tau)] \cap \beta[M - \xi(t)] \neq \emptyset\},$$

where $\gamma(t,\tau)$ is Pontryagin's selection which is measurable in τ, $\tau \in [0,t]$. Its support function in the direction +1 is

$$\beta^*(t,\tau) = sup\{\beta : \beta \in \mathrm{B}(t,\tau)\},\ t \geq \tau \geq 0.$$

Since $0 \in \mathrm{B}(t,\tau)$ for $(t,\tau) \in \Delta$, by virtue of the Pontryagin condition, then

$$\beta_*(t,\tau) = \inf\{\beta : \beta \in \mathrm{B}(t,\tau)\} = 0,\ t \geq \tau \geq 0.$$

If $\xi(t)\bar{\in}M$, then, by the theorem on characterization and inverse image [29], the mapping $\mathrm{B}(t,\tau)$ is closed-valued and measurable in τ, $\tau \in [0,t]$. Correspondingly, on the basis of the theorem on support function [29], the function $\beta^*(t,\tau)$ is also measurable in τ, $\tau \in [0,t]$. In the case $\xi(t) \in M$ for some $t > 0$

$$\mathrm{B}(t,\tau) = [0,+\infty),\ \tau \in [0,t],$$

and $\beta^*(t,\tau) = +\infty$ for all $\tau \in [0,t]$.

The resolving function $\beta^*(t,\tau)$ generates the guaranteed time

$$p_1(g(\cdot),\gamma(\cdot,\cdot)) = \inf\{t \geq 0 : \int_0^t \beta^*(t,\tau)d\tau \geq 1\}. \tag{19}$$

Theorem 5 *Let for conflict-controlled process (3)–(4) the Pontryagin condition hold, $M = \mathrm{co}M$ and let for the function $g(\cdot)$ and the Pontryagin selection $\gamma(\cdot,\cdot)$*

$$p_1 = p_1(g(\cdot),\gamma(\cdot,\cdot)) < +\infty,$$

with infinum in the relation (19) attained.

Then a trajectory of the process (3) can be brought into the terminal set (4) at the time instant p_1 by some counter-control, therewith

$$\inf_{\gamma(\cdot,\cdot)\in\Gamma} p_1(g(\cdot),\gamma(\cdot,\cdot)) = p(g(\cdot)). \tag{20}$$

Proof of the first part of statement follows from the proof of Theorem 2, and the equality (20) is obtained in the case of differential games in [34, Theorem 6]. In the case under study the proof is similar.

From the above provided constructions, under assumptions of Theorem 1, 2, it readily follows the inequality

$$\inf_{\gamma(\cdot,\cdot)\in\Gamma} t(g(\cdot),\gamma(\cdot,\cdot)) \leq p(g(\cdot)). \tag{21}$$

The case of equality (21) is highlighted by the following statement.

Theorem 6 *Let for the conflict-controlled process (3), (4) Pontryagin's condition hold and let us suppose that for some shift function the mapping $\mathsf{A}(t,\tau,v)$, $(t,\tau) \in \Delta$, $v \in V$, is convex-valued, besides set M is convex.*

Then, if for some T, $T \in T(g(\cdot),\gamma(\cdot,\cdot)) \neq \emptyset$ the condition

$$\bigcap_{v\in V}[W(T,\tau,v) - \alpha(T,\tau)M] = W(T,\tau) - \alpha(T,\tau)M,\ 0 \leq \tau \leq T, \tag{22}$$

holds, then

$$\inf_{\gamma(\cdot,\cdot)} t(g(\cdot),\gamma(\cdot,\cdot)) = \inf_{\gamma(\cdot,\cdot)} \delta(g(\cdot),\gamma(\cdot,\cdot)) = p(g(\cdot)).$$

Proof It will suffice to prove the inequality

$$p(g(\cdot)) \leq \inf_{\gamma(\cdot,\cdot)\in\Gamma} t(g(\cdot),\gamma(\cdot,\cdot)). \tag{23}$$

Since $\alpha(T,\tau) \in \mathsf{A}(T,\tau,v)$, $(t,\tau) \in \Delta$, $v \in V$, then, by virtue of the property for convex-valuedness of the mapping $\mathsf{A}(T,\tau,v)$, we have from the relation (9)

$$[W(T,\tau,v) - \gamma(T,\tau)] \cap \alpha(T,\tau)[M - \xi(T)] \neq \emptyset,\ 0 \leq \tau \leq T,\ v \in V.$$

The latter is equivalent to the inclusion

$$0 \in W(T,\tau,v) - \alpha(T,\tau)M + \alpha(T,\tau)\xi(T) - \gamma(T,\tau),\ \tau \in [0,T],\ v \in V,$$

or

$$0 \in \bigcap_{v\in V}[W(T,\tau,v) - \alpha(T,\tau)M] + \alpha(T,\tau)\xi(T) - \gamma(T,\tau),\ \tau \in [0,T].$$

Taking into account the condition (22) we get

$$0 \in W(T,\tau) - \alpha(T,\tau)M + \alpha(T,\tau)\xi(T) - \gamma(T,\tau),\ \tau \in [0,T].$$

Upon integration from 0 to T both parts of this inclusion, taking into account the condition $M = \mathrm{co}M$, we have

$$\pi g(T) \in M - \int_0^T W(T,\tau)d\tau.$$

This implies the inequality (23) and, therefore, the theorem is proved.

Applications of approach methods for conflict-controlled processes to design of control systems are presented, for example, in [13, 14].

Illustrative example

Dynamics of conflict-controlled process is defined by the system of equations

$$\dot{z} = \lambda z + u - v, z \in R^n, z(0) = z_0, \lambda \in R^n, \tag{24}$$

and control parameters are subject to the constraints

$$\|u\| \le a \ge 1, \quad \|v\| \le 1.$$

This means that in the general scheme $A = \lambda E$, E- unit quadratic matrix of order n, control block is $\varphi(u, v) = u - v$, and control domains are $U = aS = S_a$, $V = S$, where $S = \{z : \|z\| \le 1\}$ is a unit ball centered at the origin.

Terminal set is given by the relationship $\|z\| \le \varepsilon$. It presents itself the ε—neighbourhood of the origin and in the case of separated motions of the players, corresponds to the approach of the pursuer with the evader at a distance of ε. Hence,

$$M^* = M = \varepsilon S, M_0 = \{0\}.$$

The orthogonal complement L to M_0 in R^n is all space R^n, $L = R^n$, and the operator of orthogonal projecting $\pi : R^n \to L$, is the operator of identical transformation, given by the unit matrix E. The fundamental matrix $e^{At} = e^{\lambda t}E$. Then the set-valued mapping $W(t) = e^{\lambda t}[aS \overset{*}{-} S] = e^{\lambda t}(a-1)S \neq \emptyset$.

Since $a \ge 1$ Pontryagin's condition holds and W(t) is a ball of radius $e^{\lambda t}(a-1)$ centered at the origin. The time, provided by the first direct method, is determined by the inclusion

$$e^{\lambda t} z_0 \in \varepsilon S - \int_0^t e^{\lambda t}(a-1)S d\tau \tag{25}$$

In view of the properties of Aumann integral [30] of set-valued mapping with spherical images,

$$\int_0^t e^{\lambda \tau}(a-1)S d\tau = \begin{cases} \frac{1}{\lambda}(e^{\lambda t} - 1)(a-1)S, \lambda \neq 0, \\ t(a-1)S, \lambda = 0. \end{cases}$$

Since the images enjoy the property for central symmetry, the minus sign before integral in (25) must be replaced by the plus sign. Then the inclusion (25) takes the form ($\lambda \neq 0$)

$$e^{\lambda t} z_0 \in [\varepsilon + \frac{1}{\lambda}(e^{\lambda t} - 1)(a-1)]S. \tag{26}$$

Of particular interest is the least time, at which the inclusion (26) holds. By virtue of continuity of the left-hand part and the radius of ball in the right-hand part of inclusion (26), this time is determined by the equation

$$\|e^{\gamma t} z_0\| = \varepsilon + \frac{1}{\lambda}(e^{\lambda t} - 1)(a-1), \tag{27}$$

that corresponds to the first instant of the vector $e^{\lambda t} z_0$ hitting the boundary of the ball $[\varepsilon + \frac{1}{\lambda}(e^{\lambda t} - 1)(a-1)]S$. From Eq. (27) we deduce that

$$e^{\lambda t} = \frac{\varepsilon - \frac{1}{\lambda}(a-1)}{\|z_0\| - \frac{1}{\lambda}(a-1)} = \frac{\lambda\varepsilon - (a-1)}{\lambda\|z_o\| - (a-1)},$$

hence

$$t = \frac{1}{\lambda}\ln\frac{\lambda\varepsilon - (a-1)}{\lambda\|z_0\| - (a-1)} = \min\{t > 0 : t \in P(z_0)\} \tag{28}$$

Since $\|z_0\| > \varepsilon$, then this time is positive only for $\lambda < 0$. Even in the case of equal resources ($a = 1$) for $\lambda < 0$ the time (28) is finite if $\varepsilon \neq 0$. If, otherwise $\varepsilon = 0$, then the trajectory hits the origin only at the time $+\infty$. Note that for $\lambda < 0$ the constraint $0 < \frac{\lambda\varepsilon - (a-1)}{\lambda\|z_0\| - (a-1)} < 1$ should hold that imposes condition that the numerator and the denominator should be of the same sign. Because the denominator is negative, the numerator should be negative too: $\lambda\varepsilon - (a-1) < 0$. But this is always the case if $\lambda < 0$.

If $\lambda = 0$ from (25) there follows the inclusion

$$z_0 \in \varepsilon S + t(a-1)S = [\varepsilon + t(a-1)]S.$$

The first moment of hitting is defined by the equation $\|z_0\| = \varepsilon + t(a-1)$, whence we have $t = \frac{\|z_0\| - \varepsilon}{a-1}$.

Note, that when $\lambda > 0$, in order time t be positive the inequality in (28) $\frac{\lambda\varepsilon - (a-1)}{\lambda\|z_0\| - (a-1)} > 1$ should be fulfilled that leads to the inequality $\varepsilon > \|z_0\|$, that contradicts the initial assumption $\|z_0\| > \varepsilon$. Thus, in this case it is impossible to terminate the game (24) in a finite time. The selection of Pontryagin's mapping $W(t, \tau)$ has the form

$$\gamma(t, \tau) = -e^{\lambda(t-\tau)}(a-1)\frac{z_0}{\|z_0\|}.$$

It defines the counter-control of the pursuer

$$u(\tau) = v(\tau) - (a-1)\frac{z_0}{\|z_0\|},$$

which is performing parallel ε-approach at the moment t in the point $\varepsilon\frac{z_0}{\|z_0\|}$.

In fact, upon substitution this control into representation of the trajectory in the form of Cauchy formula we obtain

$$z(t) = e^{\lambda t} z_0 - \int_0^t e^{\lambda t}(a-1)\frac{z_0}{\|z_0\|}d\tau$$

$$= \frac{\lambda\varepsilon - (a-1)}{\lambda\|z_0\| - (a-1)} z_0 - \frac{1}{\lambda}\left(\frac{\lambda\varepsilon - (a-1)}{\lambda\|z_0\| - (a-1)} - 1\right)(a-1)\frac{z_0}{\|z_0\|}$$

$$= \frac{z_0}{\|z_0\|}\left[\frac{\lambda\varepsilon - (a-1)}{\lambda\|z_0\| - (a-1)}\|z_0\| - \frac{1}{\lambda}\left(\frac{\lambda\varepsilon - \lambda\|z_0\|}{\lambda\|z_0\| - (a-1)}\right)\right](a-1)$$

$$= \frac{z_0}{\|z_0\|}\left[\frac{(\lambda\varepsilon - (a-1))\|z_0\| - (\varepsilon - \|z_0\|)(a-1)}{\lambda\|z_0\| - (a-1)}\right]$$

$$= \frac{z_0}{\|z_0\|}\frac{\varepsilon(\lambda\|z_0\| - (a-1))}{\lambda\|z_0\| - (a-1)} = \varepsilon\frac{z_0}{\|z_0\|}.$$

Parallelism of the ε- approach follows from the trajectory representation

$$z(t) = [e^{\lambda t} + \int_0^t e^{\lambda\tau}\frac{a-1}{\|z_0\|}d\tau]z_0, \forall t \geq 0.$$

Let us apply the apparatus of resolving functions to solving example (24). Since the mapping $W(t,\tau)$ has the form of a ball centered at the zero then Pontryagin's condition is fulfilled. That is why, we choose in $W(t,\tau)$ a selection coinciding with the origin, i.e. $\gamma(t) \equiv 0$. Then $\xi(t) = e^{\lambda t}z_0$, the set-valued mapping A has the form

$$A(t,\tau,v) = \{\alpha \geq 0 : e^{\lambda(t-\tau)}(aS - v) \cap \alpha[\varepsilon S - e^{\lambda t}z_0] \neq \emptyset\},$$

and the resolving function

$$\alpha(t,\tau,v) = \sup\{\alpha \geq 0 : ae^{\lambda t}z_0 - e^{\lambda(t-\tau)}v \in [\alpha\varepsilon + e^{\lambda(t-\tau)}a]S\},$$

i.e. $A(t,\tau,v) = [0, \alpha(t,\tau,v)]$ is interval on the half-axis R_+.

The resolving function satisfies the equality

$$\|\alpha e^{\lambda t}z_0 - e^{\lambda(t-\tau)}v\| = \alpha\varepsilon + e^{\lambda(t-\tau)}a,$$

that can be reduced to the quadratic equation for α,

$$(\alpha e^{\lambda t}z_0 - e^{\lambda(t-\tau)}v, \alpha e^{\lambda t}z_0 - e^{\lambda(t-\tau)}v) = [\alpha\varepsilon + e^{\lambda(t-\tau)}a]^2,$$

or, in conventional form,

$$[e^{2\lambda t}\|z_0\|^2 - \varepsilon^2]\alpha^2 - 2[(e^{\lambda t}z_0, e^{\lambda(t-\tau)}v) + \varepsilon a e^{\lambda(t-\tau)}]\alpha + e^{2\lambda(t-\tau)}(\|v\|^2 - a^2) = 0.$$

Its greatest positive root appears as the resolving function

$$\alpha(t,\tau,v) = \frac{(e^{\lambda t}z_0, e^{\lambda(t-\tau)}v)}{e^{2\lambda t}\|z_0\|^2 - \varepsilon^2} + \frac{\sqrt{[(e^{\lambda t}z_0, e^{\lambda(t-v)}v]^2 + (e^{2\lambda t}\|z_0\|^2 - \varepsilon^2)e^{2\lambda(t-\tau)}(a^2 - \|v\|^2)}}{e^{2\lambda t}\|z_0\|^2 - \varepsilon^2}.$$

Taking into account that $\min_{\|v\|\leq 1}\alpha(t,\tau,v)$ is attained at $v = -\frac{z_0}{\|z_0\|}$, we obain

$$\min_{\|v\|\leq 1}\alpha(t,\tau,v) = \frac{-e^{\lambda(2t-\tau)}\|z_0\| + \varepsilon a e^{\lambda(t-\tau)} + |e^{\lambda(2t-\tau)}\|z_0\|a - \varepsilon e^{\lambda(t-\tau)}|}{e^{2\lambda t}\|z_0\|^2 - \varepsilon^2}. \quad (29)$$

Let us consider the function $e^{\lambda t}\|z_0\| - \varepsilon = f(t)$—for $\lambda < 0$. It is evident that $f(0) > 0$. As time grows this function is decreasing and turns into zero at $\bar{t} = \frac{1}{\lambda}\ln\frac{\varepsilon}{\|z_0\|} > 0$.

This time guarantees the game termination, if $a = 1$, according to the first direct method, with $\alpha(t,\tau,v) \equiv +\infty$. On the interval $[0,\bar{t}]$ $e^{\lambda t}\|z_0\|a - \varepsilon > 0$ if $a > 1$, that is why, removing the modulo sing in (29), we obtain

$$\min_{\|v\|\leq 1}\alpha(t,\tau,v) = \frac{e^{\lambda(2t-\tau)}\|z_0\|(a-1) + \varepsilon e^{\lambda(t-\tau)}(a-1)}{e^{2\lambda t}\|z_0\|^2 - \varepsilon^2} = \frac{(a-1)e^{\lambda(t-\tau)}}{e^{\lambda t}\|z_0\| - \varepsilon}$$

In view of the above function continuity in t. Hence, the minimal time of game termination, according to the method of resolving functions, is defined by the relationship

$$\int_0^t (a-1)e^{\lambda\tau}d\tau = e^{\lambda t}\|z_0\| - \varepsilon.$$

Then

$$\frac{a-1}{\lambda}(e^{\lambda t} - 1) = e^{\lambda t}\|z_0\| - \varepsilon, e^{\lambda t} = \frac{\lambda\varepsilon - (a-1)}{\lambda\|z_0\| - (a-1)},$$

$$t = \frac{1}{\lambda}\ln\frac{\lambda\varepsilon - (a-1)}{\lambda\|z_0\| - (a-1)} = t(z_0, 0). \quad (30)$$

For $\lambda < 0$ it is positive, for $\lambda = 0$ $t = \frac{\|z_0\| - \varepsilon}{a-1}$, just like in Pontryagin's first direct method. Taking into account the expression for $B(t,\tau,v)$, we find the targeting points in the set εS and through them the control of pursuer in an explicit form. Let us consider the set-valued mapping

$$\mathrm{M}(t,\tau,v)=\{m\in\varepsilon S:\alpha(t,\tau,v)(m-e^{\lambda t}z_0)\in e^{\lambda(t-\tau)}(aS-v)\},$$
$$t\geq\tau\geq 0,\ v\in S.$$

In this example it consists of a unique selection

$$m(t,\tau,v)=\varepsilon\frac{\alpha(t,\tau,v)e^{\lambda t}z_0-e^{\lambda(t-\tau)}v}{\|\alpha(t,\tau,v)e^{\lambda t}z_0-e^{\lambda(t-\tau)}v\|}. \tag{31}$$

In fact, the set-valued mappings $\alpha(t,\tau,v)(\varepsilon S-e^{\lambda t}z_0)$ and $e^{\lambda(t-\tau)}(aS-v)$ are closed and convex-valued. That their images intersect means that their difference contains zero. We evaluate the support functions of the left-hand and the right-hand part of this inclusion and obtain following inequality

$$\alpha(t,\tau,v)\varepsilon\|p\|+(e^{\lambda(t-\tau)}v-e^{\lambda t}\alpha(t,\tau,v)z_0,p)+e^{\lambda(t-\tau)}a\|p\|\geq 0,$$
$$\forall p,\ \|p\|=1. \tag{32}$$

Minimum in p of its left-hand side is attained at the element

$$p=\frac{e^{\lambda t}\alpha(t,\tau,v)z_0-e^{\lambda(t-\tau)}v}{\|e^{\lambda t}\alpha(t,\tau,v)z_0-e^{\lambda(t-\tau)}v\|}.$$

From this it follows that the selection of the mapping $\mathrm{M}(t,\tau,v)$, at which maximum of support function of mapping $\alpha(t,\tau,v)\varepsilon S$ is attained, with account of the inequality (32), has the form (31).

Then the control, realizing the guaranteed time (30), has the form

$$u(\tau)=e^{-\lambda(T-\tau)}\{e^{\lambda(T-\tau)}v(\tau)+\bar{\alpha}(T,\tau,v(\tau))[m(T,\tau,v(\tau))-e^{\lambda T}z_0]\},$$

$$T=t(z_0,0),$$

where

$$\bar{\alpha}(T,\tau,v(\tau))=\begin{cases}\alpha(T,\tau,v),\ \tau\in[0,t_*),\\ 0,\ \tau\in[t_*,T],\end{cases}$$

and

$$m(T,\tau,v)=\varepsilon\frac{\bar{\alpha}(T,\tau,v)e^{\lambda T}z_0-e^{\lambda(T-\tau)}v}{\|\bar{\alpha}(T,\tau,v)e^{\lambda T}z_0-e^{\lambda(T-\tau)}v\|}.$$

Here the time t_* divides the active and passive parts and appears as a root of the testing function

$$h(t)=1-\int_0^t\alpha(T,\tau,v(\tau))d\tau,\ t>0.$$

Since there exists a moment of switching, then a prehistory of control of the evader is used and, generally, the pursier applies control prescribed by a quasi-strategy. This control appears as counter-control both on the active and passive parts.

On other hand, because the mapping $\mathrm{A}(t,\tau,v)$ is convex-valued, then the game can be terminated in the class of counter-controls. Really, since

$$\alpha(T)=\int_0^T\inf_{\|v\|\leq 1}\alpha(T,\tau,v)d\tau=\int_0^T\frac{(a-1)e^{\lambda(T-\tau)}}{e^{\lambda T}\|z_0\|-\varepsilon}d\tau=1.$$

then

$$\alpha^*(T,\tau)=\inf_{\|v\|\leq 1}\alpha(T,\tau,v)=\frac{(a-1)e^{\lambda(T-\tau)}}{e^{\lambda T}\|z_0\|-\varepsilon}.$$

Thus, there does not exist a moment of switching if the testing function is treated in the form

$$h_0(t)=1-\int_0^t\alpha^*(T,\tau)d\tau,$$

because $h_0(T)=1$.

Control of the pursuer has the form

$$u(\tau)=e^{-\lambda(T-\tau)}\{e^{\lambda(T-\tau)}v(\tau)+\alpha^*(T,\tau)[m_1(T,\tau,v(\tau))-e^{-\lambda T}z_0]\},$$

where

$$m_1(T,\tau,v)=\varepsilon\frac{\alpha^*(T,\tau)e^{\lambda T}z_0-e^{\lambda(T-\tau)}v}{\|\alpha^*(T,\tau)e^{\lambda T}z_0-e^{\lambda(T-\tau)}v\|}.$$

It is easy to see that

$$\mathrm{A}(t,\tau)=\bigcap_{v\in S}\mathrm{A}(t,\tau,v)=[0,\alpha(t,\tau)]=[0,\alpha^*(t,\tau)],$$

since

$$\alpha(t,\tau)=\alpha^*(t,\tau)=\frac{(a-1)e^{\lambda(t-\tau)}}{e^{\lambda t}\|z_0\|-\varepsilon}$$

and, thus, Theorem 2 is illustrated by the above provided calculations.

Now we proceed to analysis of the method of resolving functions with fixed points of the set εS. As before, selection $\gamma(t) \equiv 0$. Then $\eta(t, m) = e^{\lambda t} z_0 - m$ and the set-valued mapping

$$\mathrm{A}(t, \tau, v, m) = \{\alpha \geq 0 : -\alpha(e^{\lambda t} z_0 - m) \in e^{\lambda(t-\tau)}(aS - v)\}.$$

The support function of its images in direction +1, (the resolving function) satisfies the equality

$$\|e^{\lambda(t-\tau)} v - \alpha(m - e^{\lambda t} z_0)\| = e^{\lambda(t-\tau)} a,$$

that implies the corresponding quadratic equation for α. Its greatest positive root has the form

$$\alpha(t, \tau, v, m) = \frac{(e^{\lambda(t-\tau)} v, e^{\lambda t} z_0 - m)}{\|e^{\lambda t} z_0 - m\|^2} + \frac{\sqrt{((e^{\lambda(t-\tau)} v, e^{\lambda t} z_0 - m)^2 + \|e^{\lambda t} - m\|^2 e^{2\lambda(t-\tau)}(a^2 - \|v\|^2)}}{\|e^{\lambda t} z_0 - m\|^2}.$$

From this it follows that

$$\min_{\|v\| \leq 1} \alpha(t, \tau, v, m) = \frac{e^{\lambda(t-\tau)}(a-1)}{\|e^{\lambda t} z_0 - m\|},$$

with minimum attained at the element $v = -\frac{e^{\lambda t} z_0 - m}{\|e^{\lambda t} z_0 - m\|}$.

Then, in this scheme the least time of the game termination is

$$t(z_0, 0) = \min\{t > 0 :$$

$$\max_{m \in \varepsilon S} \int_0^t \frac{e^{\lambda(t-\tau)}(a-1)}{\|e^{\lambda t} z_0 - m\|} d\tau = 1\} = \min\{t > 0 : \frac{\int_0^t e^{\lambda(t-\tau)}(a-1) d\tau}{\min_{m \in \varepsilon S} \|e^{\lambda t} z_0 - m\|} = 1\}.$$

Minimum in the denominator is attained at the element $m = \varepsilon \frac{z_0}{\|z_0\|}$.

Hitting the given point m will occur at the moment

$$t_m = \min\{t > 0 : \frac{1}{\lambda}(e^{\lambda t} - 1)(a - 1) = \|e^{\lambda t} z_0 - m\|\},$$

with the help of the control

$$u(\tau) = e^{-\lambda(t_m - \tau)}\{e^{\lambda(t_m - \tau)} v(\tau) - \bar{\alpha}(t_m, \tau, v(\tau), m)\eta(t_m, m)\},$$

where

$$\bar{\alpha}(t, \tau, v, m) = \begin{cases} \alpha(t_m, \tau, v, m), \tau \in [0, t_m^*), \\ 0, \tau \in [t_m^*, t_m], \end{cases}$$

and t_m^* is a zero of the testing function $h_m(t) = 1 - \int_0^t \alpha(t_m, \tau, v(\tau), m) d\tau$.

Now we examine the case in example (24), when the control domain U presents itself a boundary of the ball S_a, i.e. $U = \partial S_a$. Then Pontryagin's condition does not hold because

$$W(t, \tau) = e^{\lambda(t-\tau)}[\partial S_a \underline{*} S] = \emptyset.$$

Let us set the shift function $\gamma(t, \tau)$ equal to zero, $\xi_0(t) = e^{\lambda t} z_0$.

The set-valued mapping

$$\mathrm{A}(t, \tau, v) = \{\alpha \geq 0 : e^{\lambda(t-\tau)}(\partial S_a - v) \cap \alpha[\varepsilon S - e^{\lambda t} z_0] \neq \emptyset\}$$

has non-empty images when $t \geq \tau \geq 0$, $v \in S$ and therefore Condition 3 is fulfilled.

From geometric considerations it is easy to see that the upper resolving function of the first type $\alpha^*(t, \tau, v)$ coincides with the resolving function $\alpha(t, \tau, v)$ and, therefore, the moment of the game termination is

$$T = T(z_0, 0) = \begin{cases} \frac{1}{\lambda} \ln \frac{\lambda\varepsilon - (a-1)}{\lambda\|z_0\| - (a-1)}, \lambda < 0 \\ \frac{\|z_0\| - \varepsilon}{a-1}, \lambda = 0 \end{cases} \tag{33}$$

Let us deduce formula for the resolving function of first type:

$$\begin{aligned} \alpha_*(t, \tau, v) &= \inf\{\alpha \geq 0 : e^{\lambda(t-\tau)}(\partial S_a - v) \cap \alpha[\varepsilon S - e^{\lambda t} z_0] \neq \emptyset \\ &= \sup\{\alpha \geq 0 : \alpha[\varepsilon S - e^{\lambda t} z_0] \subset e^{\lambda(t-\tau)}(aS - v)\} \\ &= \sup\{\alpha \geq 0 : \|\alpha e^{\lambda t} z_0 - e^{\lambda(t-\tau)} v\| = e^{\lambda(t-\tau)} \cdot a - \alpha\varepsilon\} \end{aligned}$$

The greatest positive root of corresponding quadratic equation

$$\begin{aligned} &[e^{2\lambda t}\|z_0\|^2 - \varepsilon^2]\alpha^2 - 2[(e^{\lambda t} z_0, e^{\lambda(t-\tau)} v) - \varepsilon a e^{\lambda(t-\tau)}]\alpha \\ &+ e^{2\lambda(t-\tau)}(\|v\|^2 - a^2) = 0 \end{aligned}$$

has the form

$$\alpha_*(t, \tau, v) = \frac{(e^{\lambda t} z_0, e^{\lambda(t-\tau)} v) - \varepsilon a e^{\lambda(t-\tau)}}{e^{2\lambda t}\|z_0\|^2 - \varepsilon^2}$$

$$+\frac{\sqrt{[(e^{\lambda t}z_0, e^{\lambda(t-\upsilon)}v) - \varepsilon a e^{\lambda(t-\tau)}]^2 + (e^{2\lambda t}\|z_0\|^2 - \varepsilon^2)e^{2\lambda(t-\tau)}(a^2 - \|v\|^2)}}{e^{2\lambda t}\|z_0\|^2 - \varepsilon^2}.$$

Then
$\mathrm{A}(t,\tau,v) = [\alpha_*(t,\tau,v), \alpha^*(t,\tau,v)]$ and $0 \notin \mathrm{A}(t,\tau,v)$.
The equality $\alpha_*(t,\tau,v) = 0$ means that Pountryagin's condition is fulfilled with

$$\sup_{\|v\|\le 1} \alpha_*(t,\tau,v) = \frac{(a+1)e^{\lambda(t-\tau)}}{e^{\lambda t}\|z_0\| + \varepsilon},$$

attained at $v = \frac{z_0}{\|z_0\|}$. Then

$$\mathrm{A}(t,\tau) = \bigcap_{\|v\|\le 1} \mathrm{A}(t,\tau,v) = [\frac{(a+1)e^{\lambda(t-\tau)}}{e^{\lambda t}\|z_0\| + \varepsilon}, \frac{(a-1)e^{\lambda(t-\tau)}}{e^{\lambda t}\|z_0\| - \varepsilon}]$$
$$= e^{\lambda(t-\tau)}[\frac{a+1}{e^{\lambda t}\|z_0\| + \varepsilon}, \frac{a-1}{e^{\lambda t}\|z_0\| - \varepsilon}]$$

and Condition 3, implying that $\mathrm{A}(t,\tau) \neq \emptyset,\ t \ge \tau \ge 0$, is fulfilled if $\frac{(a+1)}{e^{\lambda t}\|z_0\|+\varepsilon} \le \frac{(a-1)}{e^{\lambda t}\|z_0\|-\varepsilon}$, that leads to the inequality $e^{\lambda t}\|z_0\| - a\varepsilon \le 0$.

In this case the upper and the lower resolving functions of second type are

$$\alpha^*(t,\tau) = \frac{e^{\lambda(t-\tau)}(a-1)}{e^{\lambda t}\|z_0\| - \varepsilon}, \quad \alpha_*(t,\tau) = \frac{e^{\lambda(t-\tau)}(a+1)}{e^{\lambda t}\|z_0\| + \varepsilon}.$$

Since $\alpha^*(T) = 1$ and in case $e^{\lambda T}\|z_0\| - a\varepsilon < 0$,

$$\alpha_*(T,\tau) < \alpha^*(T,\tau),$$

then $\alpha_*(T) < 1$ and from Theorem 4 there follows that the game can be terminated in the time T, given by the Eq. (33).

This process is realized with help of the pursuer control

$$u(\tau) = e^{-\lambda(T-\tau)}\{e^{\lambda(T-\tau)}v(\tau) + \approx{\alpha}(T,\tau,v(\tau))[m(T,\tau,v(\tau)) - e^{\lambda T}z_0]\}, \quad (34)$$

where

$$\tilde{\tilde{\alpha}}(T,\tau,v) = \begin{cases} \alpha(T,\tau,v), \tau \in [0, t_*^1), \\ \alpha_*(T,\tau), \tau \in [t_*^1, T], \end{cases}.$$

and

$$m(T,\tau,v) = \varepsilon\frac{\tilde{\tilde{\alpha}}(T,\tau,v)e^{\lambda T}z_0 - e^{\lambda(T-\tau)}v}{\|\tilde{\tilde{\alpha}}(T,\tau,v)e^{\lambda T}z_0 - e^{\lambda(T-\tau)}v\|}.$$

The moment of switching t_*^1 is a zero of testing function

$$h(t) = 1 - \int_0^t \alpha^*(T,\tau,v(\tau))d\tau - \int_t^T \alpha_*(T,\tau)d\tau, t \in [0,T].$$

The functions $\alpha^*(T,\tau,v)$ and $\alpha_*(T,\tau)$ are found above in analytic form. Since $h^1(T) = 1$, then the approach can be realized with help of the control of the form (7,11) with function $m(T,\tau,v)$, in which $\tilde{\tilde{\alpha}}(T,\tau,v)$ is replaced for function $\alpha^1(T,\tau)$, $\tau \in [0,T]$.

In Theorem 3 the inequality (11) is provided by the relationship

$$e^{\lambda T}\|z_0\| - a\varepsilon \le 0.$$

Guaranteed control of the pursuer is constructed analogously to the previous case and, therefore, in this case

$$u(\tau) = e^{-\lambda(T-\tau)}\{e^{\lambda(T-\tau)}v(\tau) + \alpha(T,\tau)[m(T,\tau,v(\tau)) - e^{\lambda T}z_0]\},$$

where

$$m(T,\tau,v) = \varepsilon\frac{\alpha(T,\tau)e^{\lambda T}z_0 - e^{\lambda(T-\tau)}v}{\|\alpha(T,\tau)e^{\lambda T}z_0 - e^{\lambda(T-\tau)}v\|}, \tau \in [0,T].$$

7 Conclusions

Various schemes of the method of resolving functions for investigation of the quasi-linear game dynamic problems are presented in the paper. In so doing, the case, when the classic Pontryagin's condition does not hold, is analyzed. To tackle this difficulty, we take into consideration the upper and the lower resolving functions of two types. This makes it feasible to derive sufficient conditions for approach a cylindrical terminal set by a trajectory of conflict-controlled process in the class of quasi- and stroboscopic strategies. That a solution of dynamic system is presented in a rather general form allows encompass in unified form a wide spectrum of functional-differential systems. A comparison of guaranteed times of the schemes of the first direct method and the method of resolving functions is made.

Analytical results are exemplified by the model example with simple matrix and spherical control domains.

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